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THE GROWTH OF POWERS OF A NON-NEGATIVE MATRIX

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THE GROWTH OF POWERS OF A NON-NEGATIVE MATRIX

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ABSTRACT

Let A be a non-negative $n \times n$ matrix. In this paper ~~we study~~ the growth of the powers A^m, $m = 1,2,3,\ldots$ is studied. These powers occur naturally in ~~the~~ an iteration process

A to the m power

$$x^{(m+1)} = A\,x^{(m)},\ x^{(0)} \geq 0$$

which is important in applications and numerical techniques. Roughly speaking, ~~we analyze~~ the asymptotic behavior of each entry of A^m * ~~We apply our~~ The main result is applied to determine necessary and sufficient conditions for the convergence to the spectral radius of A of certain ratios naturally associated with the iteration above.

* is analyzed.

AMS(MOS) Subject Classification: 15A18, 15A29, 15A48, 65F10

Key Words: Non-negative matrices, powers, reduced graph, singular distance, final state, iteration

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Significance and Explanation

The matrix problem considered in this paper arises when studying the iteration process

$$x^{(m+1)} = A\, x^{(m)} \qquad (*)$$

where A and $x^{(0)}$ are non-negative. This situation occurs often in mathematical economics, population genetics, numerical techniques and in other fields.

The convergence of the above iterative procedure depends on the size of the largest eigenvalue of A (the spectral radius, $\rho(A)$). In practice it is laborious to compute the spectral radius directly, and approximate methods for estimating the spectral radius are important.

Using the notation of (*), let

$$r_m(x) = \min_i \frac{(x^{(m+1)})_i}{(x^{(m)})_i}, \quad R_m(x) = \max_i \frac{(x^{(m+1)})_i}{(x^{(m)})_i}.$$

This paper determines necessary and sufficient conditions for $r_m(x)$ and $R_m(x)$ to tend to the spectral radius $\rho(A)$ of A. It is then easy to compute $\rho(A)$ in practice. The above problem is completely analyzed by determining the asymptotic behaviour of each entry of A^m.

For any non-negative x denote $r(x) = \min_i (Ax)_i/x_i$ and $R(x) = \max_i (Ax)_i/x_i$. We determine necessary and sufficient conditions in terms of the reduced graph of A such that

$$\lim_{m\to\infty} r(A^m x) = \underset{m\to\infty}{R}(A^m x) = \rho(A).$$

This is important for numerical procedure for calculating $\rho(A)$ - the spectral radius of A.

The responsibility for the wording and views expressed in this descriptive summary lies with MRC, and not with the authors of this report.

THE GROWTH OF POWERS OF A NON-NEGATIVE MATRIX

Shmuel Friedland and Hans Schneider

1. Introduction.

Let A be a non-negative $n \times n$ matrix. In the iteration process

$$(1.1) \qquad x^{(m+1)} = Ax^{(m)}, \quad x^{(0)} \geq 0,$$

which is important in applications and numerical techniques, the powers A^m, $m = 1,2,3,\ldots$ occur naturally. In this paper we study the growth of these powers. When A is irreducible or stochastic, the behavior of A^m is well studied, e.g. Gantmacher [7, Ch. 13, §5-7], Varga [17, pp. 32-34]. In these cases, the elementary divisors belonging to the spectral radius $\rho(A)$ of A are linear. We deal here with the general non-negative case, when the elementary divisors belonging to $\rho(A)$ may have degrees greater than 1. At the cost of ignoring nilpotent A, where the problem is trivial, we assume that $\rho(A) = 1$.

For a complex $n \times n$ matrix A, with $\rho(A) = 1$, there is a least integer k for which $m^{-k} A^m$ is bounded, $m = 1,2,3,\ldots$. However, even in the simple case of an imprimitive, irreducible non-negative A, $\lim_{m\to\infty} \|m^{-k} A^m\|$ and, a fortiori, $\lim_{m\to\infty} m^{-k} A^m$, do not in general exist. To obtain precise results for general non-negative A with $\rho(A) = 1$ it is thus necessary to introduce some smoothing. For example, in [13] Rothblum considered Cesaro means of powers of A. In this paper we study the growth of

$$(1.2) \qquad B^{(m)} = A^m (I + \ldots + A^{q-1}), \quad m = 1,2,\ldots$$

where q is a certain positive integer.

After some preliminaries in §2, we use elementary analytic methods in §3 to prove a theorem on the growth of $B^{(m)}$. As corollary, we obtain a known theorem on the index of the eigenvalue 1 of A, cf. Schaefer [15, Ch. 1, Thm. 2.7]. We also give a local form of the theorem, that is, we show that for $1 \leq i, j \leq n$ there exist integers $k = k(i,j)$ and $q = q(i,j) > 0$ such that the element $b_{ij}^{(m)}$ of the matrix given by (1.2) satisfies

$$(1.3) \qquad \lim_{m\to\infty} m^{-k} b_{ij}^{(m)} > 0 .$$

The analytic results of §3 motivate the investigations in the rest of the paper.

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The main thrust of the paper is the use of the graph structure of the matrix A to decrease the integer $q(i,j)$ and to determine the integer $k(i,j)$ in (1.3). The requisite graph theoretic concepts are developed in §4 and in §5 we state our main result, Theorem (5.10). As a corollary, we obtain a striking theorem on the index of 1 due to Rothblum [12].

In §6 we apply our results to the iteration process (1.1). For $x \geq 0$, $x \neq 0$ denote

(1.4.i) $\qquad r(x) = \sup\{\mu : \mu x \leq Ax\}$,

(1.4.ii) $\qquad R(x) = \inf\{\rho : \rho x \geq Ax\}$.

In Theorem (6.8) we find necessary and sufficient conditions for $r(A^m x)$ and $R(A^m x)$ to converge to the spectral radius of A. In §7, we show that a theorem due to D. H. Carlson [3] on the existence of non-negative solutions y for $(I - A)y = x$, $x \geq 0$ is a consequence of our main results and we extend the theorem.

2. Preliminaries.

(2.1) Notations.

Let $\varphi(1), \varphi(2), \ldots,$ be a sequence of non-negative numbers and $k \geq 0$ be an integer.

(i) $\varphi(m) = O(m^k)$

will denote that $\varphi(m)/m^k$, $m = 1,2,\ldots,$ is bounded.

(ii) $\varphi(m) = o(m^k)$

will denote that $\lim_{m \to \infty} \varphi(m)/m^k = 0$.

(iii) $\varphi(m) \approx m^k$

will denote that $\lim_{m \to \infty} \varphi(m)/m^k$ exists and is positive.

(iv) The above notations will also be used for $k = -1, -\infty$. In case that $k = -1$ $\varphi(m) = O(m^k)$, $\varphi(m) = o(m^k)$, $\varphi(m) \approx m^k$ will each indicate that there exists ρ, $0 < \rho < 1$, such that $\varphi(m)\rho^{-m} = O(1)$. In case that $k = -\infty$ the above notations will mean that $\varphi(m) = 0$ for all sufficiently large m. (Thus $\varphi(m) \approx m^{-\infty}$ implies $\varphi(m) \approx m^{-1}$.)

(v) The notation $A(m) \approx m^k$ will be used for a sequence of non-negative matrices $A(1), A(2), \ldots$ to indicate the relation holds for each element.

Combinatorial Result.

Let $r \geq 0$ and $t > 0$ be integers. Then

$$\Gamma_t^r = \sum_{p_1+\cdots+p_t=r} 1^{p_1} 1^{p_2} \cdots 1^{p_t} \tag{2.2}$$

where the summation is taken over all non-negative integers $p_1, \ldots, p_t$ whose sum is r. It is well known that

$$\Gamma_t^r = \binom{r+t-1}{r} . \tag{2.3}$$

The simplest way to prove this equality is by considering the coefficient of x^r of both sides of the identity

$$\sum_{r=0}^{\infty} \binom{r+t-1}{r} x^r = \left(\sum_{r=0}^{\infty} x^r\right)^{-t}$$

which is derived from $(1-x)^{-t} = (1-x)^{-1} \ldots (1-x)^{-1}$. For a purely combinatorial proof see for example Brualdi [2, p. 37]. For $t = 0$ the above formula implies $\Gamma_0^r = 1$ for all $r \geq 0$.

We shall also need some results on the convergence of series.

(2.4) <u>Lemma</u>: Let $k \geq 1$ and let $b_p \geq 0$, $p = 0,1,2,\ldots$ be a sequence such that

$$p^{-(k-1)} \lim_{p\to\infty} (b_p + \ldots + b_{p+q-1}) = v , \tag{2.5}$$

where $q > 0$. Then

$$m^{-k} \lim_{m\to\infty} \sum_{p=1}^{m} b_p = \frac{v}{kq} . \tag{2.6}$$

<u>Proof</u>: Elementary. Alternatively, check that $c_{m,p} = m^{-k} k p^{k-1}$ satisfies the assumptions of Hardy [8, Theorem 2, p. 43].

(2.7) <u>Lemma</u>: Suppose (2.5) holds. If $\lim_{m\to\infty} a_m = u$ then

$$m^{-k} \lim_{m\to\infty} \sum_{p=1}^{m} a_p b_{m-p} = \frac{uv}{kq} . \tag{2.8}$$

<u>Proof</u>: According to Hardy [8, Theorem 16, p. 64]

$$\lim_{m\to\infty} \frac{\sum_{p=1}^{m} a_p b_{m-p}}{\sum_{p=1}^{m} b_p} = u \tag{2.9}$$

since

$$0 \leq \frac{b_m}{\sum_{p=1}^{m} b_p} \leq \frac{b_m + \ldots + b_{m+q-1}}{\sum_{p=1}^{m} b_p} \leq \frac{2\, v\, m^{(k-1)}}{v(2kq)^{-1} m^k} ,$$

and the last expression tends to 0. If we apply (2.6) to (2.9) we obtain (2.8). ∎

3. Analytic approach.

By $\mathbb{R}$, resp. $\mathbb{C}$, we denote the real, resp. complex field, and by $\mathbb{R}_+$ the non-negative numbers. The set of real, resp. complex, non-negative $r \times n$ matrices will be denoted by $\mathbb{R}^{rn}$, resp. $\mathbb{C}^{rn}$, $\mathbb{R}_+^{rn}$. We also write $A \geq 0$ for $A \in \mathbb{R}_+^{rn}$ (A is non-negative) and $A > 0$ when A is positive ($a_{ij} > 0$, $i = 1,\dots,r$, $j = 1,\dots,n$).

Let $A \in \mathbb{C}^{nn}$. By spec A we denote the set of eigenvalues of A. Suppose that spec $A = \{\lambda_1,\dots,\lambda_r\}$, where the λ_α are pairwise distinct. It is known, cf. Gantmacher [7, Ch. 5, §3], that there exist non-negative integers $p_1,\dots,p_r$ and unique matrices $Z^{(\alpha\beta)} \in \mathbb{C}^{nn}$, $\beta = 0,\dots,p_\alpha$, $\alpha = 1,\dots,r$ which are linearly independent such that for each polynomial $f(\tau)$

$$f(A) = \sum_{\alpha=1}^{r} \sum_{\beta=0}^{p_\alpha} f^{(\beta)}(\lambda_\alpha) Z^{(\alpha\beta)} . \tag{3.1}$$

The $Z^{(\alpha\beta)}$ are polynomials in A, $p_\alpha + 1$ is the size of a largest Jordan-block belonging to λ_α. The columns of $Z^{(\alpha p_\alpha)}$ are eigenvectors of A corresponding to the eigenvalue λ_α, the rank of $Z^{\alpha p_\alpha}$ is equal to the number Jordan blocks of size $p_\alpha + 1$ corresponding to λ_α. (The simplest way to obtain (3.1) is by assuming that A is in Jordan form). As usual we define

$$\operatorname{index}(\lambda_\alpha) = p_\alpha + 1.$$

That is $p_\alpha + 1$ is the multiplicity of λ_α in the minimal polynomial of A. We shall also use a localized index. For $1 \leq i,j \leq n$ we put

$$\operatorname{index}_{ij}(\lambda_\alpha) = 1 + \max\{\beta : z_{ij}^{(\alpha\beta)} \neq 0 ,\ \beta = 0,\dots,p\} ,$$

where $\operatorname{index}_{ij}(\lambda_\alpha) = 0$ if $z_{ij}^{(\alpha\beta)} = 0$, $\beta = 0,\dots,p_\alpha$. If $A \in \mathbb{C}^{nn}$ and m is any integer we shall denote the elements of A^m by $a_{ij}^{(m)}$, $1 \leq i,j \leq m$.

Let $A \in \mathbb{R}_+^{nn}$. We assume throughout the normalization $\rho(A) = 1$. It is well-known (Frobenius [6], Gantmacher [7, Ch. 13], Berman-Plemmons [1, Ch. 2]) that if λ is an eigenvalue of A and $|\lambda| = 1$, then λ is a root of 1. Hence there is a positive integer q such that $\lambda^q = 1$, for all $\lambda \in$ spec A, $|\lambda| = 1$. The smallest such integer q will be called the <u>period</u> of A. If $q = 1$, A will be called <u>aperiodic</u>. For an irreducible and aperiodic matrix $A \geq 0$ the Frobenius theorem and the formula (3.1) imply

$$\lim_{m\to\infty} A^m = Z^{(10)} > 0$$

where $\lambda_1 = 1$, see for example Berman-Plemmons [1, Thm. 4.1]. Theorem (3.4) extends the above equality in a local way. Part (i) of the theorem is an extension of the known inequality apparently due to Schaefer [14, p. 264, Thm. 2.4]

(3.2) $$\operatorname{index}(\lambda) \le \operatorname{index}(1) \ , \ \text{if} \ |\lambda| = 1 \ ,$$

for non-negative matrices, see also Schaefer [15, Ch. 1, Thm. 2.7], Berman-Plemmons [1, Thm. 3.2]. This result and Part (i) of Theorem (3.4) could easily be deduced from the classical Pringsheim theorem functions, e.g. Titchmarsh [16, p. 214]. The use of the Pringsheim theorem in analyzing the spectral properties of non-negative matrices can be traced back to Ostrowski [10], see also Karlin [9] and Schaefer [14, App.] for the infinite dimensional case. See Friedland [5] for a detailed analysis of the Pringsheim theorem for rational functions which has certain analogs of the Frobenius theorem. For sake of compelteness we bring a short and elementary independent proof of Theorem (3.4). To do so we need an easy lemma which probably is known.

(3.3) __Lemma__. Let λ_α, z_α, $\alpha = 1,\ldots,r$ be complex numbers, where the λ_α are pairwise distinct. If $\lim_{m\to\infty} (\sum_{\alpha=1}^{r} \lambda_\alpha^m z_\alpha)$ exists, then $z_\alpha = 0$ if $|\lambda_\alpha| \ge 1$, $\lambda_\alpha \ne 1$.

__Proof__: Since $\lim_{m\to\infty} \lambda_\alpha^m$ exists for $|\lambda_\alpha| < 1$ or $\lambda_\alpha = 1$, without loss of generality we may assume that $|\lambda_\alpha| \ge 1$, $\lambda_\alpha \ne 1$, $\alpha = 1,\ldots,r$. Put $z = (z_1,\ldots,z_r)^t \in \mathbb{C}^n$, and $u^{(m)} = (u_m,\ldots,u_{m+r-1})^t$, where $u_m = \sum_{\alpha=1}^{r} \lambda_\alpha^m z$. Let $\Lambda = \operatorname{diag}(\lambda_1,\ldots,\lambda_r) \in \mathbb{C}^{rr}$ and let $V = (v_{\alpha\beta})_1^r \in \mathbb{C}^{rr}$ be the Vandermond matrix given by $v_{\alpha\beta} = \lambda_\beta^{\alpha-1}$, $\alpha,\beta = 1,\ldots,r$. Then

$$u^{(m)} = V\Lambda^m z \ .$$

The assumption of the lemma implies that $\lim_{m\to\infty} u^{(m)}$ exists. Since V is non-singular, $\lim_{m\to\infty} \Lambda^m z = \lim_{m\to\infty} V^{-1}u^{(m)}$ and so $z = 0$. ∎

(3.4) <u>Theorem</u>. Let $A \in \mathbb{R}_+^{nn}$ where $\rho(A) = 1$. Let $1 \le i, j \le n$.

(i) If $\lambda \in \operatorname{spec} A$, $|\lambda| = 1$, then $\operatorname{index}_{ij}(\lambda) \le \operatorname{index}_{ij}(1)$.

(ii) Let q be a positive integer such that $\lambda^q = 1$ if $\lambda \in \operatorname{spec} A$, $|\lambda| = 1$ and let $k + 1 = \operatorname{index}_{ij}(1)$. Let

$$B^{(m)} = A^m(I + \dots + A^{q-1}).$$

Then $b_{ij}^{(m)} \approx m^k$. In particular, $a_{ij}^{(m)} \neq o(m^k)$, if $k \ge 0$.

<u>Proof</u>: (i) Let $\{\lambda_1,\dots,\lambda_r\}$ be the eigenvalues with $|\lambda_\alpha| = 1$, $\alpha = 1,\dots,r$, where the λ_α are pairwise distinct. Let

$$d + 1 = \max\{\operatorname{index}_{ij}(\lambda_\alpha): \ \alpha = 1,\dots,r\}.$$

If $d = -1$ then there is nothing to prove. So assume that $d \ge 0$.
Suppose that $z_\alpha \equiv z_{ij}^{(\alpha d)} \neq 0$ for $\alpha = 1,\dots,s$ where $1 \le s \le r$. It follows immediately from (3.1) that

$$a_{ij}^{(m)} = m^d\left(\sum_{\alpha=1}^{s} \lambda_\alpha^{m-d} z_\alpha\right) + o(m^d).$$

Hence, by Lemma (3.3), $a_{ij}^{(m)} \neq o(m^d)$.

Let q be a positive integer such that $\lambda_\alpha^q = 1$, $\alpha = 1,\dots,s$. Define

$$\psi_m(\tau) = \tau^m(1 + \tau + \dots + \tau^{q-1}).$$

If we take the d-th derivative of $\psi_m(\tau)$, we obtain

$$\psi_m^{(d)}(\tau) = m^d\psi_{m-d}(\tau) + o(m^d),$$

for any fixed τ, $|\tau| \le 1$, and also $\psi_{m-d}(\lambda_\alpha) = 0$ for $|\lambda_\alpha|=1$ $\lambda_\alpha \neq 1$, $1 \le \alpha \le s$. Put $B^{(m)} = \psi_m(A)$. By (3.1) and the equality above we have

$$b_{ij}^{(m)} = m^d\left(\sum_{\alpha=1}^{r} \psi_{m-d}(\lambda_\alpha)z_\alpha\right) + o(m^d). \tag{3.5}$$

Now suppose that $\operatorname{index}_{ij}(1) < d + 1$. Then (3.5) implies that $b_{ij}^{(m)} = o(m^d)$. But $b_{ij}^{(m)} = a_{ij}^{(m)} + \dots + a_{ij}^{(m+q-1)} \ge a_{ij}^{(m)} \ge 0$ and this is a contradiction. Thus $d=k$ and this proves (i).

(ii) Suppose that $\lambda_1 = 1$. By (3.5) and the preceeding argument we obtain

$$b_{ij}^{(m)} = m^k q z_1 + o(m^k),$$

where $z_1 = z_{ij}^{k} > 0$. This proves (ii). ∎

We now state a global version of Theorem (3.4), part (ii) which follows immediately from Theorem (3.4).

(3.6) <u>Theorem</u>. Let $A \in \mathbb{R}_+^{nn}$ where $\rho(A) = 1$. Let q be a positive integer such that $\lambda^q = 1$ if $\lambda \in \operatorname{spec} A$, $|\lambda| = 1$ and $k + 1 = \operatorname{index}(\lambda) = \operatorname{index}(1)$. Let

$$B^{(m)} = A^m(I + \dots + A^{q-1}) .$$

Then

$$\lim_{m \to \infty} m^{-k} B^{(m)} = F ,$$

where $F \geq 0$ and F is not identically zero. ∎

4. Graph theoretical concepts.

Let $A \in R_+^{nn}$ and let $\rho(A) = 1$. We may assume, without loss of generality, that after simultaneous permutations of rows and columns, A is in the Frobenius [6] normal form which can be found in many references e.g. Gantmacher [7, Vol. II, p. 75]. Thus

$$(4.1) \qquad A = \begin{bmatrix} A_{11} & A_{12} & \cdots & A_{1\nu} \\ & A_{22} & & \\ & & \ddots & \\ 0 & & & A_{\nu\nu} \end{bmatrix}$$

where the diagonal blocks $A_{\alpha\alpha}$, $\alpha = 1,\ldots,\nu$, are irreducible and all subdiagonal blocks are 0. (The 1×1 matrix 0 is considered to be irreducible).

Let A be in Frobenius normal form (4.1). Then the (reduced) graph $G(A)$ of A is a subset of $\langle\nu\rangle \times \langle\nu\rangle$, where $\langle\nu\rangle = \{1,\ldots,\nu\}$ and $G(A) = \{(\alpha,\beta) \in \langle\nu\rangle \times \langle\nu\rangle : A_{\alpha\beta} \neq 0\}$. (Observe that many authors would call $G(A)$ the arcset of the graph $(\langle\nu\rangle, G(A))$, but we have no need to mention the vertex set $\langle\nu\rangle$ explicitly).

If $(\alpha,\beta) \in G(A)$ we call (α,β) an arc of $G(A)$. If (α,β) is an arc of $G(A)$, then $\alpha \leq \beta$ and $(\alpha,\alpha) \in G(A)$, $1 \leq \alpha \leq \nu$, unless $A_{\alpha\alpha}$ is the 1×1 matrix 0. Thus we define a (simple) path from α to β in $G(A)$ to be a sequence $\pi = (\alpha_0,\ldots,\alpha_s)$, where either $s \geq 1$, $1 \leq \alpha = \alpha_0 < \ldots < \alpha_s = \beta \leq \nu$ and $(\alpha_{i-1},\alpha_i) \in G(A)$, $i = 0,\ldots,s-1$, or $s = 0$ and $\alpha = \alpha_0 = \beta$ and $(\alpha,\alpha) \in G(A)$. The support of π is the set $\operatorname{supp}\pi = \{\alpha_0,\ldots,\alpha_s\} \subseteq \{1,\ldots,\nu\}$. We always assume that the α_i, $i = 0,\ldots,s$ have been listed in strictly ascending order.

If $1 \leq \alpha \leq \nu$, then we call α a singular vertex (of $G(A)$) if $\rho(A_{\alpha\alpha}) = 1$. (This terminology is consistent with that of Richman-Schneider [11]). Let $1 \leq \alpha \leq \beta \leq \nu$. For any path π from α to β let $k(\pi) + 1$ be the number of singular α in the support of π. (Thus note each distinct α is counted only once in $k(\pi) + 1$). Let $\alpha_{j_0} < \alpha_{j_1} < \ldots < \alpha_{j_k}$, where $k = k(\pi)$, be all singular vertices in $\operatorname{supp}\pi$. If there is no path from α to β in $G(A)$ we put $k(\pi) = -\infty$. Next we set

$$(4.2) \qquad k(\alpha,\beta) = \max\{k(\pi) : \pi \text{ is a path from } \alpha \text{ to } \beta \text{ in } G(A)\}.$$

We shall call $k(\alpha,\beta)$ the <u>singular</u> <u>distance</u> from α to β. If (i,i) is a position in $A_{\alpha\alpha}$ and (j,j) a position in $A_{\beta\beta}$ then we shall also call $k[i,j]$ the singular distance from i to j (note our use of square brackets).

A path π from α to β will be called a <u>maximal</u> <u>path</u> if the number of singular vertices in the support of π is $k(\alpha,\beta)+1$. Let $1 \leq \alpha, \beta \leq \nu$. Let $P(a,b)$ be the set of maximal paths from α to β. For each $\pi \in P(\alpha,\beta)$ let $q(\pi)$ be the g.c.d. of periods of $A_{\gamma\gamma}$ with $\gamma \in \operatorname{supp} \pi$ and singular (viz. $\rho(A_{\gamma\gamma}) = 1$).

Then we define

(4.3) $$q(\alpha,\beta) = \text{l.c.m } q(\pi):\pi \in P(\alpha,\beta)\}.$$

We shall call $q(\alpha,\beta)$ the <u>local</u> <u>period</u> of (α,β). If $k(\alpha,\beta) < 0$ then $q(\alpha,\beta) = 1$. Also if (i,i) is a position in $A_{\alpha\alpha}$ and (j,j) is a position in $A_{\beta\beta}$ then we shall put $q(\alpha,\beta) = q[i,j]$, the local period of (i,j).

5. The main results.

Let $A \in \mathbb{R}_+^{nn}$, where $\rho(A) = 1$, be in Frobenius normal form (4.1). It follows from the Perron-Frobenius theory for non-negative matrices, e.g., Gantmacher [7, Ch. 13] that there is a diagonal matrix X with positive diagonal elements so that, upon replacing A by XAX^{-1}

(5.1) $$A_{\alpha\alpha} = \rho(A_{\alpha\alpha})A'_{\alpha\alpha} ,$$

where $A'_{\alpha\alpha}$ is a stochastic matrix,

(5.2) $$\|A_{\alpha\beta}\|_\infty \le \sigma, \quad 1 \le \alpha < \beta \le \nu ,$$

where $1 > \sigma > \max\{\rho(A_{\alpha\alpha}) : \rho(A_{\alpha\alpha}) < 1 , \alpha = 1,\dots,\nu\}$. Here $\|\ \|_\infty$ is the ℓ_∞-operator norm,

$$\|Z\|_\infty = \max\{ \sum_{j=1}^{n} |z_{ij}| : i = 1,\dots,r\} \quad \text{for } Z \in \mathbb{R}^{rn} .$$

The diagonal matrix X can be constructed as follows. Let $u^{(\alpha)}$ be a positive vector satisfying $A_{\alpha\alpha}u^{(\alpha)} = \rho(A_{\alpha\alpha})u^{(\alpha)}$. Denote by X_α a diagonal matrix, whose diagonal entries are the elements of $u^{(\alpha)}$. Then X is of the form $\mathrm{diag}(X_1, \epsilon X_2,\dots,\epsilon^{\nu-1}X_\nu)$ for some small enough positive ϵ. In our subsequent proofs we may assume that A has been normalized as above.

Let π be a path in $G(A)$. Denote by $s+1$ the cardinality of $\mathrm{supp}\,\pi$. That is

(5.3.i) $$\mathrm{supp}\,\pi = \{\beta_0,\dots,\beta_s\} , \quad 1 \le \beta_0 < \beta_2 < \dots < \beta_s \le \nu.$$

We define the <u>path</u> <u>matrix</u> $A(\pi)$ by

(5.3.ii) $$\begin{cases} A_{ii}(\pi) = A_{\beta_i\beta_i} & i = 0,\dots,s \\ A_{i,i+1}(\pi) = A_{\beta_i\beta_{i+1}} & i = 0,\dots,s-1 \\ A_{ij}(\pi) = 0, \text{ otherwise} & i,j = 0,\dots,s, \end{cases}$$

(5.3.iii) $$A(\pi) = (A_{ij}(\pi))_0^s .$$

Thus $A(\pi)$ is in Frobenius normal form and has $s+1$ irreducible diagonal blocks $A_{ii}(\pi) = A_{\beta_i\beta_i}$, $i = 0,\dots,s$. To avoid ambiguity, we write $A(\pi)_{ij}^{(m)}$ for the (i,j) block component of $A(\pi)^m$, $i,j = 0,\dots,s$.

We now prove a sequence of lemmas for the path matrix $A(\pi)$ of a given path.

(5.4) __Lemma.__ Let $A \in \mathbb{R}_+^{nn}$ where $\rho(A) = 1$. Let $1 \leq \alpha, \beta \leq \nu$ and π be a path in $G(A)$ from α to β. Put $k = k(\pi)$, where $k(\pi) + 1$ is the number of singular vertices in supp π. If $A(\pi)$ is the path matrix given by (5.3), then $\|A(\pi)_{0s}^{(m)}\|_\infty = O(m^k)$.

__Proof:__ We note that

$$(5.5) \qquad A(\pi)_{0s}^{(m)} = \sum_{p_0+\dots+p_s=m-s} A_{00}^{p_0}(\pi)A_{01}(\pi)A_{11}^{p_1}(\pi)\dots A_{(s-1)s}(\pi)A_{ss}^{p_s}(\pi) .$$

So

$$\|A(\pi)_{0s}^{(m)}\|_\infty \leq \sigma^s \sum_{p_0+\dots+p_s=m-s} \|A_{00}(\pi)\|_\infty^{p_0} \dots \|A_{ss}(\pi)\|_\infty^{p_s} .$$

Suppose first that π does not contain singular vertices, i.e. $k = -1$. Then

$$\|A(\pi)_{0s}^{(m)}\|_\infty \leq \sigma^m \sum_{p_0+\dots+p_s=m-s} 1^{p_0}\dots 1^{p_s} = \sigma^m \Gamma_s^{m-s},$$

where Γ_t^r is given by (2.3). As $\Gamma_s^{m-s} \leq m^s$ we immediately deduce

$$\lim_{m\to\infty} \tau^{-m}A(\pi)_{0s}^{m} = 0 , \quad \text{for any } \tau, \sigma < \tau < 1 .$$

Suppose now that $k \geq 0$. Then

$$\|A(\pi)_{0s}^{(m)}\|_\infty \leq \sigma^s \sum_{q_0+\dots+q_s=m-s} 1^{q_0}\dots 1^{q_k}\sigma^{q_{k+1}}\dots\sigma^{q_s} =$$

$$\sigma^s \sum_{u=0}^{m-s} \Big(\sum_{q_0+\dots+q_k=u} 1^{q_0}\dots 1^{q_k}\Big)\Big(\sum_{q_{k+1}+\dots+q_s=m-s-u} \sigma^{q_{k+1}}\dots\sigma^{q_s}\Big)$$

$$= \sigma^s \sum_{u=0}^{m-s} \Gamma_{k+1}^{u}\,\Gamma_{s-k}^{m-s-u}\,\sigma^{m-u-s} .$$

Hence

$$\|A^{(m)}(\pi)_{0s}\|_\infty \leq \Gamma_{k+1}^{m-s}\Big(\sum_{\nu=0}^{\infty}\Gamma_{s-k}^{\nu}\,\sigma^{\nu+s}\Big) .$$

The last series converges by the ratio test and $\Gamma_{k+1}^{m-s} \leq m^k$. This establishes the lemma. ∎

(5.6) __Lemma.__ Let the assumptions of Lemma 5.4 hold. Assume furthermore that $k \geq 0$, i.e. the support of π contains singular vertices. Then, for sufficiently large m

$$(5.7) \qquad \sum_{j=0}^{2(s+1)(n-1)} A(\pi)_{0s}^{(m+j)} \geq Gm^k ,$$

where G is a positive matrix.

Proof: Let

$$B_{ii}(\pi) = I + A_{ii}(\pi) + \ldots + A_{ii}(\pi)^{(n-1)}, \quad i = 1,\ldots,s.$$

Since $A_{ii}(\pi)$ is irreducible, and its dimension does not exceed n, we have $B_{ii}(\pi) > 0$, Wielandt [18], Berman-Plemmons [1, Ch. 2, Thm. 1.3]. Clearly (5.5) implies, for $t = 2(s+1)(n-1)$

$$\sum_{j=0}^{t} A(\pi)_{0s}^{(m+j)} \geq \sum_{p_0+\ldots+p_s=m-s} B_{00}(\pi)A_{00}(\pi)^{p_0}B_{00}(\pi)A_{01}(\pi)B_{11}(\pi)A_{11}^{p_1}B_{11}(\pi) \ldots$$

$$A_{s-1,s}(\pi)\ B_{ss}(\pi)\ A_{ss}^{p_s}(\pi)B_{ss}(\pi)\ .$$

For $i,j = 0,\ldots,s$ let E_{ij} be the matrix all of whose entries equal 1 and whose dimension is that of $A_{ij}(\pi)$. Clearly $B_{00}(\pi) \geq c_0'E_{00}$, $B_{ss}(\pi) \geq c_s'E_{ss}$ where $c_0', c_s' > 0$. Since $A_{i,s+1}(\pi) \neq 0$, we have

$$B_{ii}(\pi)A_{i,i+1}(\pi)B_{i+1,i+1}(\pi) \geq C_i\ E_{i,i+1}$$

where $c_i > 0$, $i = 1,\ldots,s-1$, and hence for some $c > 0$,

$$\sum_{j=1}^{t} A(\pi)_{0s}^{(m+j)} \geq c \sum_{p_0+\ldots+p_s=m-s} E_{00}A_{00}(\pi)^{p_0}E_0,\ldots,E_{s-1,s}A_{ss}(\pi)^{p_s}E_{ss}\ . \tag{5.8}$$

In the inequality (5.8) we may restrict the sum on the right hand side by letting $p_j = 0$ if $\rho(A_{jj}) = 0$. So let $\gamma_0 < \ldots < \gamma_k$ be the subscripts of $B_{\gamma i}$ which are singular vertices and put $\bar{A}_{ii} = A_{\gamma_i\gamma_i}(\pi)$. Since $E_{ij}E_{jk} \geq E_{ik}$, it follows that

$$\sum_{j=0}^{t} A(\pi)_{0s}^{(m+j)} \geq c' \sum_{p_0+\ldots+p_k=m-s} \bar{E}_{-1,0}\bar{A}_{00}(\pi)^{p_0}\bar{E}_{01}\ldots \bar{A}_{kk}^{p_k}\ \bar{E}_{k,k+1}\bar{A}_{00}\bar{E}_{01}$$

where $c' > 0$ and the $\bar{E}_{i,i+1}$ $i = -1,\ldots,k$ are matrices all of whose entries are 1. But $\bar{A}_{ii}(\pi)$ is a stochastic matrix, $i = 0,\ldots,k$, whence $\bar{A}_{ii}(\pi)^{p_0}\bar{E}_{i,i+1} = \bar{E}_{i,i+1}$, $i = 0,\ldots,k$. It follows that

$$\sum_{j=0}^{t} A(\pi)_{0s}^{(m+j)} \geq 2\Gamma_k^{m-s}\ G\ ,$$

where $G > 0$. The lemma now follows from (2.3), since $\Gamma_k^{m-s} \geq \frac{1}{2} m^k$, for sufficiently large m. ∎

For $1 \leq \alpha,\beta \leq \nu$, we write (again without confusion) $\mathrm{index}_{(\alpha,\beta)}(1) = \max\{\mathrm{index}_{i,j}(1)\}$ as i,j ranges over all positions (i,j) in $A_{\alpha\beta}$.

(5.9) Lemma. Let the assumptions of Lemma (5.4) hold. Let $q(\pi)$ be the g.c.d. of periods of $A_{\alpha\alpha}$ for singular $\alpha \in \operatorname{supp} \pi$. Let

$$B(\pi)^{(m)} = A(\pi)^m(I + A(\pi) + \ldots + A(\pi)^{q-1}),$$

$q = q(\pi)$. Then

(i) In $A(\pi)$, $\operatorname{index}_{0,s}(1) = k(\pi) + 1$,

(ii) $\lim\limits_{k \to \infty} m^{-k} B(\pi)^{(m)}_{0,s} > 0$.

Proof: Let $\kappa + 1 = \operatorname{index}_{0,s}(1)$ in $A(\pi)$. By Theorem (3.4) there is an integer q^*, $1 \le q^* \le n$, such that, for

$$C(\pi)^{(m)} = A(\pi)^{(m)}(I + A(\pi) + \ldots + A(\pi)^{q^*-1}),$$

$$\lim_{m \to \infty} m^{-\kappa} C(\pi)^{(m)}_{0s} = F_{0s} \neq 0, \quad \text{if } k \ge 0 .$$

But $\kappa > k(\pi) = k$ contradicts Lemma (5.4), and $\kappa < k$ contradicts Lemma (5.6). Hence $\kappa = k$. This proves (i).

Now let $q = q(\pi)$. If λ is in the spectrum of $A(\pi)$, $|\lambda| = 1$ and, in $A(\pi)$, $\operatorname{index}_{\alpha,\beta}(\lambda) = \kappa(\pi) + 1$, then λ must be an eigenvalue of every $A_{\alpha\alpha}$, for singular $\alpha \in \operatorname{supp}(\pi)$. Hence $\lambda^q = 1$, $q = q(\pi)$. We immediately obtain (ii) from Theorem (3.4) and Lemma (5.6). ∎

We now state our main result.

(5.10) Theorem. Let A be non-zero $n \times n$ matrix normalized by the condition $\rho(A) = 1$. Assume $1 \le i, j \le n$. Let $k = k[i,j]$ be the singular distance from i to j and $q = q[i,j]$ be the local period of (i,j). Put $B^{(m)} = A^m(I + A + \ldots + A^{q-1})$. Then $b^{(m)}_{ij} \approx m^k$.

Proof: As usual, we assume that A is in the Frobenius form (4.1). Suppose that (i,i) is a position in $A_{\alpha\alpha}$ and (j,j) a position in $A_{\beta\beta}$. Denote by $\Pi(\alpha,\beta)$ the set of all paths connecting α to β. Then we obviously have

$$A^{(m)}_{\alpha\beta} = \sum_{\pi \in \Pi(\alpha,\beta)} A(\pi)^{(m)}_{0s(\pi)} .$$

So

$$B^{(m)}_{\alpha\beta} = \sum_{\pi \in \Pi(\alpha,\beta)} B(\pi)^{(m)}_{0s(\pi)} .$$

Assume first that $k = k(\pi) = -\infty$ then clearly $B^{(m)}_{\alpha\beta} = A^{(m)}_{\alpha\beta} = 0$. If $k = -1 \ge k(\pi)$ then lemma (5.4) implies that each $A(\pi)^{(m)}_{0s(\pi)} \approx m^{-1}$. So $A^{(m)}_{\alpha\beta} \approx m^{-1}$ and again $A^{(m)}_{\alpha\beta} = B^{(m)}_{\alpha\beta}$.

Assume now that $k \geq 0$. If $k > k(\pi)$ Lemma (5.4) implies that $B^{(m)}_{0s(\pi)}(\pi) = o(m^k)$. However, if $k = k(\pi)$, then according to Lemma (5.9) $\lim_{m\to\infty} m^{-k} B(\pi)^{(m)}_{0s(\pi)} = F_{0s}(\pi) > 0$, as $q(\pi)$ divides $q(\alpha,\beta) = q[i,j]$. By the definition of $k(\alpha,\beta)$ there exists $\pi \in \Pi(\alpha,\beta)$ such that $k(\pi) = k(\alpha,\beta)$. So $\lim_{k\to\infty} m^{-k} B^{(m)}_{\alpha\beta} = F_{\alpha\beta} > 0$. ∎

(5.11) Corollary. Under the conditions of Theorem (5.10),

$$\sum_{p=1}^{m} a^{(p)}_{ij} \approx m^{(k+1)} .$$

Proof: For $k \geq 0$, the result is immediate by Lemma (2.4). If $k = -1$, then by Theorem (5.10) the non-negative series above converges. The assumption $k = -1$ implies that at least one term is positive. Finally if $k = -\infty$, $a^{(p)}_{ij} = 0$, $p = 1,2,\ldots$, and the result follows. ∎

Comparing Theorems 3.4 and 5.10 we first deduce a local version of Rothblum's equality and then the equality itself.

(5.12) Theorem. Let $A \in \mathbb{R}^{nn}_{+}$ where $\rho(A) = 1$. Assume that $1 \leq i, j \leq n$, then

$$\mathrm{index}_{ij}(1) = k[i,j] + 1 .$$ ∎

(5.13) Corollary. (Rothblum [12]) Let $A \in \mathbb{R}^{nn}_{+}$ where $\rho(A) = 1$. Then

$$\mathrm{index}(1) = \max_{1\leq i,j\leq n} \mathrm{index}_{ij}(1) = \max_{1\leq i,j\leq n} k[i,j] + 1.$$ ∎

Evidently our results (5.10)-(5.13) are easily modified to apply to all non-negative A with $\rho(A) > 0$.

6. Convergent iterative methods for the spectral radius of a non-negative matrix.

Let $A \in \mathbb{R}_+^{nn}$ and assume that $\rho(A) > 0$. Let $r(x)$ and $R(x)$ be defined as in (1.4). Clearly $0 \leq r(x) \leq R(x) \leq +\infty$. It is obvious that

$$r(x) \leq r(Ax) \leq r(Ax) \leq R(x) .$$

So the sequence $r(A^m x)$, $m = 0,1,\ldots,$ is an increasing sequence bounded above by $R(x)$ and the sequence $R(A^m x)$, $m = 0,1,\ldots$, is a decreasing sequence bounded below by $r(x)$.

In [4] Collatz observed that for $A \in \mathbb{R}_+^{nn}$ and $x > 0$

$$r(x) \leq \rho(A) \leq R(x) , \tag{6.1}$$

and when A is irreducible, this inequality is valid for all $x \geq 0$, $x \neq 0$, see Wielandt [18], Varga [17, p. 32]. Thus the question arises when for $A \geq 0$ and $x \geq 0$, $x \neq 0$

$$\lim_{m \to \infty} r(A^m x) = \rho(A) = \lim_{m \to \infty} R(A^m x) . \tag{6.2}$$

Wielandt's [18] characterization of $\rho(A)$ for irreducible A easily implies that (6.2) holds for primitive A and all $x \in \mathbb{R}_+^n$, $x \neq 0$, $x \geq 0$ cf. Varga [17, p. 34]. This result follows from the fact that

$$\lim_{m \to \infty} \rho(A)^{-m} A^m = Z > 0 ,$$

when A is primitive, where $Z = uv^t$, $v > 0$ $Au = \rho(A)u$, $v > 0$, $v^t A = \rho(A) v^t$, $v^t u = 1$. If A is irreducible but imprimitive then (6.2) does not hold unless x is orthogonal on all eigenvectors of A^t corresponding to λ such that $|\lambda| = \rho(A)$ and $\lambda \neq \rho(A)$. We shall show that this condition can be put in equivalent forms. If A is irreducible and of period q, then by simultaneous permutations of rows and columns we now put A into the form

$$\begin{bmatrix} 0 & A_{12} & 0 & \cdots & 0 \\ 0 & 0 & A_{23} & \cdots & 0 \\ \vdots & & & & \\ 0 & 0 & 0 & \cdots & A_{q-1,q} \\ A_{q1} & 0 & 0 & \cdots & 0 \end{bmatrix} \tag{6.3}$$

where the diagonal blocks 0 are square, Frobenius [6], Gantmacher [7, Vol II, p. 62], Berman - Plemmons [1, Ch. 2, Thm. 2.20].

(6.4) Lemma. Let A be an irreducible non-negative matrix of period q in form (6.3), and suppose that $\rho(A) = 1$. Let $v^tA = v^t$, $Au = u$, where $v > 0$, $u > 0$, $v^tu = 1$, $A^ty^j = \omega^jy^j$, $j = 1,\ldots,q-1$, $\omega = e^{2\pi i/q}$. Let $0 \neq x \in \mathbb{R}^n_+$ be partitioned conformally with A, $x^t = (x^t_{(1)},\ldots,x^t_{(q)})$. Then the following are equivalent

(i) $\lim_{m\to\infty} A^mx = (v^tx)u$,

(ii) $\lim_{m\to\infty} A^mx$ exists,

(iii) $x^ty^j = 0$, $j = 1,\ldots,q-1$,

(iv) $v^t_{(1)}x_{(1)} = \ldots = v^t_{(q)}x_{(q)}$,

(v) $\lim_{m\to\infty} R(A^mx) = \lim_{m\to\infty} r(A^mx) = 1$,

where $v^t = (v^t_{(1)},\ldots,v^t_{(q)})$ has been partitioned conformally with A.

Proof: We first derive a formula for A^mx, $m = 1,2,\ldots$. Let ω be a primitive q-th root of unity. It is well known that the eigenvalues of A on the unit circle are $\lambda_\alpha = \omega^{\alpha-1}$, $\alpha = 1,\ldots,q$ and that each λ_α is a simple zero of the characteristic polynomial. It follows, in the notation of §3, that $p_\alpha = 0$, $\alpha = 1,\ldots,q$ and that

$$Z^{(\alpha 0)} = D^{\alpha-1}uv^tD^{1-\alpha}, \quad \alpha = 1,\ldots,q$$

$$y^\alpha = D^{(1-\alpha)}v, \quad \alpha = 1,\ldots,q-1,$$

where

$$D = \begin{bmatrix} I_{11} & & & 0 \\ & \omega I_{22} & & \\ & & \ddots & \\ 0 & & & \omega^{q-1}I_{qq} \end{bmatrix}$$

and $I_{\alpha\alpha}$ is an identity matrix of the same order of $A_{\alpha\alpha}$, $\alpha = 1,\ldots,q$.

Hence by (3.1)

$$A^m = \sum_{\alpha=0}^{q-1} \omega^{m\alpha} D^\alpha uv^tD^{-\alpha} + o(1)$$

and so

$$(6.5)\qquad A^m x = \sum_{\alpha=0}^{q-1} \omega^{m\alpha} a_\alpha (D^\alpha u) + o(1)$$

where

$$(6.6)\qquad a_\alpha = v^t D^{-\alpha} x = x^T y^\alpha, \quad \alpha = 0,\dots,q-1 .$$

Let

$$c_\beta = v^t_{(\beta+1)} x_{(\beta+1)}, \quad \beta = 0,\dots,q-1 .$$

Then it follows immediately from (6.6) that

$$(6.7)\qquad a_\alpha = \sum_{\beta=0}^{q-1} \omega^{-\alpha\beta} c_\beta, \quad \alpha = 0,\dots,q-1 .$$

We now prove the equivalence of our five conditions. We show (i) $\Rightarrow$ (ii) $\Rightarrow$ (iii) $\Rightarrow$ (iv) $\Rightarrow$ (i) and (i) $\Rightarrow$ (v) $\Rightarrow$ (iv).

(i) $\Rightarrow$ (ii) Trivial.

(ii) $\Rightarrow$ (iii) Since $\lim_{m\to\infty} A^m x$ exists, $\lim_{m\to\infty} v^t D^{-\alpha} A^m x$ also exists, $\alpha = 0,\dots,q-1$. But $v^t u > 0$, and hence $a_\alpha = x^t y^\alpha = 0$, $\alpha = 1,\dots,q-1$ by Lemma (3.3).

(iii) $\Rightarrow$ (iv) Consider the identity (6.7). Since the Vandermonde matrix $q^{-1/2}(\omega^{-\alpha\beta})$, $\alpha,\beta = 0,\dots,q-1$ is unitary the assumption $a_\alpha = x^t y^\alpha = 0$, $\alpha = 1,\dots,q-1$ implies that $c_0 = c_1 = \dots = c_{q-1}$, which proves (iv).

(iv) $\Rightarrow$ (i) If (iv) hold then $c_0 = c_1 = \dots = c_{q-1}$ and (6.7) implies $a_1 = \dots = a_{q-1} = 0$. This establishes (i) in view of (6.5) and (6.6).

(i) $\Rightarrow$ (v) Trivial, since $v^t x > 0$ and $u > 0$.

(v) $\Rightarrow$ (i) Let $m = q\ell + r$, $0 \le r \le q-1$. Then (6.5) implies

$$\lim_{\ell\to\infty} A^{q\ell+r} x = \tilde{x}^{(r)}, \qquad r = 0,\dots,q-1$$

for some $\tilde{x}^{(r)} \geq 0$, $\tilde{x}^{(r)} \neq 0$. Also

$$A^r \tilde{x}^{(0)} = \tilde{x}^{(r)}, \quad r = 0,\dots,q-1, \quad A^q \tilde{x}^{(0)} = \tilde{x}^{(0)} .$$

As A^q is a direct sum of q irreducible and primitive matrices the assumption $x \geq 0$, $x \neq 0$ implies that $\lim_{\ell\to\infty} (A^q)^\ell x = \tilde{x}^{(0)} \neq 0$. Obviously $x^{(0)} \geq 0$.

Now (v) implies that

$$x^0 \le x^{(1)} = Ax^0 \le x^{(0)},$$

whence $x^{(1)} = x^{(0)}$ and thus $x^{(r)} = x^{(0)}$ for $r = 1,\dots,q-1$. So $\lim_{m\to\infty} A^m x = x^{(0)}$ and (i) follows. ∎

In what follows we give necessary and sufficient conditions on a reducible matrix A to satisfy (6.2). To do so we need a few more graph theoretical concepts.

Let G be a graph on $\langle\nu\rangle = \{1,\ldots,\nu\}$. Let J be a non-void subset of $\langle\nu\rangle$. Then $\alpha \in J$ is called a final state with respect to J if for any $\beta \neq \alpha$ and $(\alpha,\beta) \in G$, $\beta \notin J$. Denoting by $F(J)$ the set of all final states with respect to J. If $J = \langle\nu\rangle$ then α is called a final state, i.e. $(\alpha,\beta) \in G$ implies that $\beta = \alpha$. Define

$$d(\beta,J) = \max\{k(\beta,\alpha) : \alpha \in F(J)\} .$$

If $J = \langle\nu\rangle$ then write $d(\beta)$ instead of $d(\beta,\langle\nu\rangle)$.

Let $A \geq 0$ be a reducible matrix. We assume that A is in the Frobenius form (4.1).

As in §4, denote by $G(A)$ the (reduced) graph of A. Let $x \geq 0$, $x \neq 0$. Partition x conformably with A given by (4.1). That is $x^t = (x^t_{(1)},\ldots,x^t_{(\nu)})$. The support of x is the set $\operatorname{supp} x = \{\alpha_1,\ldots,\alpha_s\} \subseteq \{1,\ldots,\nu\}$ such that $x_{(i)} \neq 0$ if and only if $i \in \operatorname{supp} x$. We shall always assume that α_i, $i = 1,\ldots,s$ have been listed in strictly ascending order.

(6.8) <u>Theorem</u>

Let $A \in \mathbb{R}^{nn}_+$, $\rho(A) = 1$. Assume that A is in the Frobenius form (4.1). Moreover if A_{ii} is imprimitive then A_{ii} is the Frobenius form (6.3) Let $x \geq 0$, $x \neq 0$. Then (6.2) holds if and only if any final state α with respect to the support of x satisfies

(i) α is a singular vertex (i.e. $\rho(A_{\alpha\alpha}) = 1$),

(ii) <u>either</u> $A_{\alpha\alpha}$ is primitive <u>or</u> $A_{\alpha\alpha}$ and $x_{(\alpha)}$ satisfy the condition (iv) of Lemma (6.4).

<u>Proof</u>: First we note that

$$(A^m x)_\alpha = \sum_{\beta \in \operatorname{supp} x} A^{(m)}_{\alpha\beta}\, x_{(\beta)} . \tag{6.9}$$

Suppose that $\alpha \in F(\operatorname{supp} x)$. Then

$$(A^m x)_\alpha = A^m_{\alpha\alpha}\, x_{(\alpha)} .$$

By the definition of $R(x)$ and $r(x)$ we have

$$r(A^m x)A^m x \leq A^{m+1}x \leq R(A^m x)A^m x .$$

So

$$r(A^m x)A^m_{\alpha\alpha}\, x_{(\alpha)} \le A^{m+1}_{\alpha\alpha}\, x_{(\alpha)} \le R(A^m x)A^m_{\alpha\alpha}\, x_{(\alpha)} .$$

Hence, since $A_{\alpha\alpha}$ irreducible, by (6.1),

$$r(A^m x) \le r(A^m_{\alpha\alpha}\, x_{(\alpha)}) \le \rho(A_{\alpha\alpha}) \le R(A^m_{\alpha\alpha}\, x_{(\alpha)}) \le R(A^m x) .$$

Assume now that (6.2) holds. Then for any final state α with respect to supp x we must have

$$\lim_{m\to\infty} r(A^m_{\alpha\alpha}\, x_{(\alpha)}) = \lim_{m\to\infty} R(A^m_{\alpha\alpha}\, x_{(\alpha)}) = \rho(A_{\alpha\alpha}) = 1 .$$

So α is a singular vertex. If $A_{\alpha\alpha}$ is imprimitive then the condition (v) of Lemma 6.4 holds. Hence $A_{\alpha\alpha}$ and $x_{(\alpha)}$ satisfy (iv) of Lemma 6.4. This proves one direction of our theorem.

Assume now that if $\alpha \in F(\text{supp } x)$ then $\rho(A_{\alpha\alpha}) = 1$ and if $A_{\alpha\alpha}$ is not primitive then $A_{\alpha\alpha}$ and $x_{(\alpha)}$ satisfy the condition (iv) of Lemma 6.4.

Let $1 \le \beta \le \nu$. Let $d = d(\beta,J)$. By our assumption $d \ne -1$. If $d = -\infty$, then $(A^m x)_\beta = 0$, $m = 1,2,\ldots$. If $d \ge 0$, then

$$m^{-d}(A^m x)_\beta = m^{-d} \sum_{\alpha\in K} A^{(m)}_{\beta\alpha}\, x_\alpha + o(1)$$

where $K = \{\alpha : k(\beta,\alpha) = d\}$. Clearly $K \subseteq F(\text{supp } x)$. Thus, to show,

$$\lim_{m\to\infty} m^{-d}(A^m x)_\beta > 0 \tag{6.10}$$

it is enough to prove

$$m^{-d}\, A^{(m)}_{\beta\alpha}\, x_\alpha > 0 , \tag{6.11}$$

for $\alpha \in F(\text{supp } x)$, $k(\beta,\alpha) = d$. To prove (6.11), let D be the matrix obtained from A by setting $D_{\alpha\alpha} = 0$ and $D_{\gamma\delta} = A_{\gamma\delta}$ in all other cases, $1 \le \gamma,\delta \le \nu$. We then have

$$m^{-d} A^{(m)}_{\beta\alpha}\, x_\alpha = m^{-d} \sum_{p=0}^{m} D^{(m-p)}_{\beta\alpha}\, A^p_{\alpha\alpha}\, x_\alpha .$$

Since in D, the singular distance from β to α is $d-1$, we have by Corollary (5.11)

$$\lim_{m\to\infty} m^{-d} \sum_{p=0}^{m} D^{(m-p)}_{\beta\alpha} = U_{\beta\alpha} > 0$$

and by Lemma (6.4)

$$\lim_{p\to\infty} A^p_{\alpha\alpha}\, x_\alpha = v_\alpha > 0 .$$

It easily follows from Lemma (2.7) that

$$\lim_{m \to \infty} m^{-d} A_{\beta\alpha}^{(m)} x_\alpha = \frac{1}{d} U_{\beta\alpha} v_\alpha > 0 \, .$$

Thus, for each β, $1 \leq \beta \leq \nu$, either $(A^m x)_\beta = 0$, $m = 1,2,\ldots$ or (6.10) is satisfied. From this (6.2) follows immediately. ∎

We remark that the restriction $\rho(A) = 1$ in Theorem (6.8) may be replaced by $\rho(A) > 0$.

(6.12) <u>Corollary</u>: Let $A \in \mathbb{R}_+^{nn}$, $\rho(A) = 1$. Assume that A is the Frobenius form (4.1). Let J be an non-empty set of $\langle \nu \rangle$. Then for any $x \geq 0$ whose support is the set J, (6.2) holds if and only if for all final state α with respect to J, $\rho(A_{\alpha\alpha}) = 1$ and $A_{\alpha\alpha}$ is primitive. ∎

(6.13) <u>Corollary</u>: Let $A \in \mathbb{R}_+^{nn}$, $\rho(A) = 1$. Assume that A is in the Frobenius form 4.1.

Then for any $x \geq 0$, $x \neq 0$, (6.2) holds if and only if for each $\alpha, \rho(A_{\alpha\alpha}) = 1$ and $A_{\alpha\alpha}$ is primitive, $\alpha = 1,\ldots,\nu$. ∎

7. Non-negative solution of $(I - A)y = x$.

As an application of our results we give a simple proof of a theorem concerning non-negative solutions y of $(I - A)y = x$ for given $x \geq 0$. For $1 \leq \alpha, \beta \leq \nu$ we shall say that β has access to α in $G(A)$ if there is a path from β to α in $G(A)$, viz $k(\beta,\alpha) \geq -1$.

(7.1) Theorem Let $A \in \mathbb{R}_+^{nn}$ with $\rho(A) = 1$ and suppose that A is in the Frobenius normal form (4.1). Let $x \in \mathbb{R}_+^n$. Then the following are equivalent:

(i) There is a $y \in \mathbb{R}_+^n$ such that $(I - A)y = x$.

(ii) No singular vertex β has access in $G(A)$ to any $\alpha \in \operatorname{supp} x$.

(iii) $\lim_{m \to \infty} (I + \dots + A^m)x$ exists.

(iv) $\lim_{m \to \infty} A^m x = 0$.

Further, if (iii) holds and $y = \lim_{m \to \infty} (I + A + \dots + A^m)x$, then $(I - A)y = x$ and

(7.2) $$y_\beta = 0, \text{ if } \beta \text{ does not have access to any } \alpha \in \operatorname{supp} x,$$

(7.3) $$y_\beta > 0, \text{ if } \beta \text{ has access to some } \alpha \in \operatorname{supp} x.$$

Proof. Let $S^{(m)} = I + A + \dots + A^m$. If $1 \leq \beta \leq \nu$, then

(7.4) $$(S^{(m)}x)_\beta = \sum_{\alpha \in \operatorname{supp} x} S^{(m)}_{\beta\alpha} x_\alpha ,$$

and, by Corollary (5.11), for $k = k(\beta,\alpha) \geq -1$,

(7.5.i) $$\lim_{m \to \infty} m^{-(k+1)} S^{(m)}_{\beta\alpha} = U_{\beta\alpha} > 0 ,$$

while for $k(\beta,\alpha) = -\infty$,

(7.5.ii) $$S^{(m)}_{\beta\alpha} = U_{\beta\alpha} = 0, m = 1,2,3,\dots .$$

We shall prove (i) $\Rightarrow$ (ii) $\Rightarrow$ (iii) $\Rightarrow$ (i), (iii) $\Rightarrow$ (iv) $\Rightarrow$ (ii).

(i) $\Rightarrow$ (ii). Suppose that $(I - A)y = x$, where $y \geq 0$. Then

$$S^{(m)} x = (I - A^{m+1})y \leq y .$$

Let β be a singular vertex. If β has access to α then $k = k(\beta,\alpha) \geq 0$ and by (7.4) and (7.5)

$$y_\beta \geq (S^{(m)}x)_\beta \geq \frac{1}{2} m^{(k+1)} U_{\beta\alpha} x_\alpha$$

for large m. Hence $x_\alpha = 0$ and $\alpha \notin \operatorname{supp} x$.

(ii) $\Rightarrow$ (iii) Suppose (ii) holds and let $1 \leq \beta \leq \alpha$.

If $\alpha \in \operatorname{supp} x$, then $k = k(\beta,\alpha) = -1$, or $k = -\infty$. Hence $\lim_{m\to\infty} S^{(m)}_{\beta\alpha} x_\alpha = U_{\beta\alpha} x_\alpha$ exists, for $\alpha \in \operatorname{supp} x_\alpha$. So by (7.5) $\lim_{m\to\infty} S^{(m)} x$ exists.

(iii) $\Rightarrow$ (i) Let $y = \lim_{m\to\infty} S^{(m)} x$. Clearly $y \geq 0$. Since $A S^{(m)} x = S^{(m+1)} x - x$, y satisfies $(I - A)y = x$. This proves (i).

(iii) $\Rightarrow$ (iv). Trivial.

(iv) $\Rightarrow$ (ii). Suppose that (iv) holds but that (ii) is false. Then there exists a singular β and an $\alpha \in \operatorname{supp} x$ such that $k(\beta,\alpha) \geq 0$. Let $q = q(\beta,\alpha)$ be the local period and let $B^{(m)} = A^m(I + \ldots + A^{q-1})$. Then $\lim_{m\to\infty} B^{(m)} x = 0$. But by Theorem (5.10) for all sufficiently large m,

$$(B^{(m)}x)_\beta \geq B^{(m)}_{\beta\alpha} x_\alpha \geq c\, m^k x_\alpha$$

where $c > 0$, and $x_\alpha \neq 0$. This is a contradiction, and the implication is proved.

To complete the proof of the theorem observe that, for $y = \lim_{m\to\infty} S^{(m)} x$,

$$y_\beta = \sum_{\alpha \in \operatorname{supp} x} U_{\beta\alpha} x_\alpha$$

in view of (ii) and (7.5). Since $U_{\beta\alpha} > 0$ if β has access to α and $U_{\beta\alpha} = 0$ otherwise, we immediately obtain (7.2) and (7.3). ∎

The equivalence of conditions (i) and (ii) in Theorem (7.1) is due to D. H. Carlson [3]. We remark that Carlson also showed that if a solution y of $(I - A)y = x$ exists, then the solution satisfying (7.2) and (7.3) is unique.

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Let A be a non-negative $n \times n$ matrix. In this paper we study the growth of the powers A^m, $m = 1,2,3,\ldots$. These powers occur naturally in the iteration process

$$x^{(m+1)} = A x^{(m)}, \quad x^{(0)} \geq 0$$

which is important in applications and numerical techniques. Roughly speaking, we analyze the asymptotic behavior of each entry of A^m. We apply our main result to determine necessary and sufficient conditions for the convergence to the spectral radius of A of certain ratios naturally associated with the iteration above.

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